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CE 30125 **November 18, 2010**
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Homework Set # 6 Corrections Nov. 23, 2010

Background: Partial differentiation in a discrete context is implemented simply by holding the indices in the directions other than those being differentiated constant. We can then apply all the one-dimensional discrete differentiation formulae. Note that for a three dimensional spatial coordinate system (x,y,z), the discrete indices would be (i,j,k).

Problem 1

Write down a second order accurate approximation to the following three dimensional partial differential equation at node (i,j,k)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial z} = B(x, y, z)$$

Background: Initial value problems are solved one step at a time (going from one time level to the next time level is called *time stepping*). It involves writing a discrete algebraic form of the differential equation to connect two time levels. Starting with the functional value specified as the initial condition, you advance from one level to the next solving for one unknown at a time using the algebraic equation that you have developed.

Problem 2

Solve the problem

$$\frac{dy}{dt} = y^2 t \frac{(5-y)}{5} \qquad y(0) = 1$$

Using a forward Euler method for $0 \le t \le 4$ with time steps equal to $\Delta t = 0.2$, $\Delta t = 0.02$, and $\Delta t = 0.002$. Compare, by plotting the three solutions and comment. Develop your own MATLAB or Fortran code to do this problem.

Background. Boundary value problems can not typically be solved one step at a time and require that the equations be simultaneously solved. This involves writing a discrete algebraic form of both boundary conditions and of the differential equation at each interior node. This way we will have the same number of algebraic equations as unknowns. These equations are then solved simultaneously.

<u>Problem 3</u>

Solve the following problem using a ***second order accurate*** Finite Difference method

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{1}{6}y = 3 \qquad 0 \le x \le 10$$

$$y(0) = 12 \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{x=10} = 2$$

using a space step $\Delta x = 2.5$, $\Delta x = 0.5$, $\Delta x = 0.1$, $\Delta x = 0.001$.

a) Draw the domain showing all the nodes. Define the unknowns

b) Write the system of equations.

c) Write <u>all</u> your equations in matrix form and solve your system of equations using MATLAB.

Hint: You may want to use the sparse matrix definition command in matlab. Prior to specifying the entries of matrix **A**, you designate it as sparse by entering:

A=sparse([])

When you do this, only nonzero locations will be stored. You then solve for the system **A**x=b using

x=A \ b

Matlab will automatically select an appropriate sparse matrix solver. This will save you lots of time when the matrix is sparse!!! (which for discrete solutions to p.d.e.'s using lots of discrete points, it always is!)

d) Plot all 4 solutions on the same graph. Now assume that the $\Delta x = 0.001$ uses a grid so small that it's errors are small compared to the other three grids. (This is a reasonable assumption given the other grid sizes. If any of the other grid sizes were within a factor of 4 or so for a typical second order method, this would not be a good assumption.) Then plot the difference between each of the 3 coarse grid solutions and the $\Delta x = 0.001$ solution against x

e) Finally quantify the error for each of your 3 coarse grids by computing the root mean square error (RMS) between the computed solution at all the nodes and the $\Delta x = 0.001$ "*reference*" solution. This is computed as:

$$E_h = \sqrt{\frac{1}{N+1}\sum_{i=0}^{N}(f_{i-h} - f_{i-\Delta x=0.001})^2}$$

where f_{i-h} equals the solution at the $N+1$ nodes for spacing $h = \frac{10}{N}$ and $f_{i-\Delta x=0.001}$ equals the fine grid solution at that node. Plot on a log-log plot for your 3 grids $\Delta x = 2.5$, $\Delta x = 0.5$, $\Delta x = 0.1$, the error versus grid spacing. What conclusions can you draw?

Background. Initial-Boundary value problems are typically solved from time level to the next time level (*"time stepping" or "time marching"*) while solving for the spatial locations either explicitly (no simultaneous equations are solved) or implicitly (a system of simultaneous equations is solved). When advancing from a know time level to an unknown time level in the space-time grid of discrete points or nodes (each one has a functional value associated with it), we write the discrete form of the boundary conditions at two boundaries (for one dimensional problems) and of the differential equation at all interior nodes.

<u>Problem 4</u>

Solve the heat conduction equation using an *explicit* in time, central in space finite difference discretization.

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial T}{\partial x}\right) \qquad 0 \le x \le 50$$

i.c.'s

$$T(x) = \frac{40}{25}x \qquad 0 \le x \le 25$$

$$T(x) = 80 - \frac{40}{25}x \qquad 25 \le x \le 50$$

b.c.'s

$$T(0) = 0$$

$$T(50) = 0$$

Diffusion is defined as a constant in space and time: $D = 25$

Let's determine the temperature distribution as follows. Develop a FD discretization for the given p.d.e. at a point in space and time, (i, j), assuming constant grids in time and space; Δt, Δx as follows

- Apply a forward in time approximation for the time derivative.
- Apply a central type approximation in space at the known time level for the spatial derivative.

Prepare a Matlab or Fortran code to solve this problem for the following discretizations

(a) $\Delta x = 10$

(b) $\Delta x = 1.0$

(c) $\Delta x = 0.1$

Select time steps for all three cases which make sense (*hint: consider* ***stability*** *constraints*). Discuss how and why you selected the time steps. Plot and compare your solutions at the following times: t = 0, 1.0, 2.0, 4.0, 8.0, 10.0. Also demonstrate that if you exceed the stability limit when selecting your time step, that your solution will be **unstable.**

<u>Problem 5</u>

Solve the heat conduction equation in Problem 4 using a *Crank-Nicolson* in time, central in space finite difference discretization. Follow the same steps you did in Problem 4. Use the same time steps you used as well as larger ones and demonstrate that this scheme is unconditionally stable, but that accuracy deteriorates when too large of a time step is selected.