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CE 30125 **November 5, 2010**
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Homework #5

Background: Discrete differentiation applies the sums and differences of functional values to estimate the continuous derivatives to a given order of accuracy. The formulae are relative to the left most point for a forward difference approximation, to the right most point for a backward difference approximation and to the center point for a central difference approximation. We examined three ways of deriving these formulae:

a) *The methods of undetermined coefficients;* We simply estimate the nth order accurate approximation to the pth derivative with the sum of $p+n$ functional values ($p+n-1$ functional values for central approximations to even derivatives) divided by h^p. We then substitute in Taylor series approximations for the functional values, collect derivative terms and force the coefficients of these derivative terms to equal the values we want so that we end up with the derivative to the desired order.

b) *Using Interpolating polynomials;* Since $f(x) = g(x) + e(x)$ we can estimate the pth derivative of $f(x)$ simply by differentiating: $f^{(p)}(x) = g^{(p)}(x) + e^{(p)}(x)$. We can apply any of our methods to derive the interpolating function and its error. We then differentiate and evaluate at the left most point in order to derive a forward difference approximation, at the right most point to derive a backward difference approximation and at the center point in order to derive a central difference approximation.

c) We can apply *difference operators* (forward, central, backward); we can average forward and backward difference operators, or apply combined forward and backward operators.

The order of the leading n^{th} order error term (if it in fact dominates and we are in the so called asymptotic range) controls how fast the solution converges as space or time resolution is reduced. On a log error vs. log grid spacing plot, the error curve will have a slope equal to n. Higher order methods can be applied to help reduce the cost of the computational solutions, since higher order methods can apply larger grid spacing and therefore fewer computational points for equivalent accuracy as compared to lower order methods.

Problem 1

Use the method of undetermined coefficients to derive a fourth order accurate central difference approximation to the second derivative, $f_i^{(2)}$. Include details of the leading order error term.

Problem 2

Use a four node Newton forward interpolation formula to derive backward approximations to the third derivative. Also, derive the error associated with your formula.

Problem 3

Evaluate the following approximations for $f^{(4)}$.

a) $\dfrac{\Delta^4 f_i}{h^4}$; b) $\dfrac{\Delta^3 \nabla f_i}{h^4}$; c) $\dfrac{\nabla^2 \Delta^2 f_i}{h^4}$

Problem 4

Evaluate the first derivative of the function:

$$f(x) = 3x^5 - 4x^3 + x - 6$$

at point $x_0 = 0$ using a second order accurate forward difference approximation to the first derivative

$$f_0^{(1)} + E_1 = \frac{-f_2 + 4f_1 - 3f_0}{2h}$$

where $f_0 \equiv f(x_0)$, $f_1 \equiv f(x_1)$, $f_2 \equiv f(x_2)$, $h \equiv x_1 - x_0 = x_2 - x_1$ and $E_1 = \textit{formal truncation error} = \frac{1}{3}h^2 f_0^{(3)}$

a) Use 5 different spacings to compute the discrete approximation to the derivative: $h = 1.0$; $h = 0.5$; $h = 0.1$; $h = 0.05$; $h = 0.01$. Complete the table shown computing the discrete approximation to the derivative and the actual error, E_{Actual}. Also compute the formal truncation error, E_1.

Note that $f_2 = f(x_2) = f(2h)$, $f_1 = f(x_1) = f(h)$ and $f_0 = f(x_0) = f(0)$,

h	f_2	f_1	f_0	$\frac{-f_2 + 4f_1 - 3f_0}{2h}$	$E_{Actual} = \frac{-f_2 + 4f_1 - 3f_0}{2h} - f^{(1)}(x_0)$	E_1
1.0						
0.5						
0.1						
0.05						
0.01						

b) Plot $\ln(E_{actual})$ vs. $\ln(h)$ for your data. On the same graph plot $\ln(E_1)$ vs. $\ln(h)$. Label your graph appropriately. Comment on

i) The slopes of your error curves and how they relate to the formal truncation error E_1 associated with the second order accurate formula.

ii) Any discrepancies which exist between the E_{actual} and E_1 curves and why all points do not lie along a straight line in the E_{actual} curve.

Background: Discrete partial differentiation in multi-dimensions can be applied simply by holding the indices in other directions constant and applying the exact same discrete differentiation formulae that were developed for one-dimensional differentiation. This is directly analogues to continuous partial differentiation

Problem 5

Consider: $\nabla^2 f$

a) Draw a molecular diagram for the fourth-order accurate central difference formulation.

b) If you used this approximation for computing $\nabla^2 f_{i-1,j}$ and $\nabla^2 f_{i+2,j+1}$, which nodal values would be shared in both approximations?