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Department of Civil Engineering
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CE 30125 **September 24, 2010**
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Homework Set #3

Backgound: *Lagrange* interpolation fits a polynomial function exactly through a number of interpolation points. The degree of this polynomial is N when we are fitting N+1 functional values. We have discussed three ways of deriving *Lagrange* interpolation:

Method 1: *Power series* defines the polynomial, implements the constraints (i.e. $g(x_i) = f_i$), and solves for the unknown polynomial coefficients by solving a system of simultaneous equations. This method is conceptually simple but is expensive when the order of the interpolator is high or when dealing with multi-dimensional interpolators. The method may also be inaccurate due to the inherent ill-conditioning of the matrix that must be solved.

Method 2: Lagrange *Basis Functions* rewrites the interpolator as a product of functional values at the nodes and the Lagrange basis function associated with each node:

$$g(x) = \sum_{i=0}^{N} f_i V_i(x)$$

Each basis function $V_i(x)$ equals zero at all nodes except at the node it is associated with where it equals 1, i.e. $V_i(x_j) = \delta_{ij}$. This makes each basis function very simple to formulate since we know that the roots are at all nodes except at the node that specific basis function is associated with. These definitions always ensure that $g(x_i) = f_i$, exactly what we want. Note that we do not solve any system of equations and that each basis function is developed as a sequence of products, simply implemented in do loops. Note that the interpolator that we derive using the Basis Function approach is identical to that derived using the power series approach.

Method 3: *Newton Forward Interpolation* reformulates the interpolation function in terms of a sequence of polynomial terms mutiplied by increasingly higher order forward differences. This method has the advantage that each higher order polynomial term has the previous term embeded in it, making it quite inexpensive. In addition, to increase the order of the interpolant, simply requires adding the next term in the sequence, making the methodology hierarchal. Finally, the error is simply estimated by considering the subsequent term in the sequence. Again this method gives the identical interpolation function as the power series and Lagrange Basis function methods.

Problem 1

Consider the function:

$$f(x) = \frac{x^4 + 2x^2 + 5}{1 + 5x + 3x^2}$$

on the interval [1, 4].

a) Develop a power series based interpolator for 4 equispaced points for the given function.
   i) Evaluate $f(x_0)$, $f(x_1)$, $f(x_2)$ and $f(x_3)$ using the specified function $f(x)$.
   ii) Define the appropriate polynomial expansion, *g(x)*.
   iii) Define the constraints needed to solve for the coefficients in the generic expansion.
   iv) Solve for the coefficients and plot the interpolation formula.
   v) Plot $f(x)$ and $g(x)$ over the interval [1, 4] using increments of 0.01. Provide labels for the axes and the curves.

b) Develop a Lagrange basis function based interpolator for 4 equispaced points for the given function.

   i) Define the appropriate polynomial expansion, *g(x)*.

   ii) Express and plot $V_0(x)$, $V_1(x)$, $V_2(x)$ and $V_3(x)$ for this problem.

   iii) Write out the expression for $g(x)$ for this problem.

   iv) Plot $f(x)$ and $g(x)$ over the interval [1, 4] using increments of 0.01. Provide labels for the axes and the curves.

c) Develop a Newton Forward Interpolation based interpolator for 4 equispaced points for the given function.

   i) Write down the appropriate Newton Forward interpolator, *g(x)*.

   ii) Develop a forward difference table.

   iii) Plot $f(x)$ and $g(x)$ over the interval [1, 4] using increments of 0.01. Provide labels for the axes and the curves.

   iv) How do the *g(x)* derived in steps a, b and c compare?

d) Estimate the error associated with $g(x)$, $e(x)$ using the next term in the Newton Forward interpolation formula. Then evaluate the actual error between $f(x)$ and $g(x)$, $E(x)$ over the interval [1, 4] using increments of 0.01. Plot both $e(x)$ and $E(x)$ over the interval. Provide labels for the axes and the curves.

e) How can you use the error term in step d , to improve the Newton Forward interpolater established in step c? Plot the now 5 point fourth degree interpolator, comparing it to the 4 point cubic interpolator established in Step c.

<u>Problem 2</u>

Develop a 3 point Newton Backward interpolation formula and the associated error. Consider points f_0, f_1 and f_2 at locations x_0, x_1 *and* x_2.

a) Step 1: Develop a 4 term Taylor series expanded about x_2. The fifth term is the truncation term (this will be the next term) and will be $O(x-x_2)^4$.

b) Step 2: Use the definition of ∇f_2 to derive an expression for the first derivative evaluated at x_2 to third order.

c) Step 3: Use the definition of $\nabla^2 f_2$ to derive an expression second derivative evaluated at x_2 to second order.

d) Step 4: Substitute into the general Taylor series in Step 1, the expressions you derived for the first and second derivatives evaluated at x_2. Sort out the terms so that a sequence of higher order terms appears and gather all the appropriate terms in your error term.

CE 30125 Computational Methods: Assignment 3

October 11, 2010

Problem 1

For this problem we consider the following function:

$$f(x) = \frac{x^4 + 2x^2 + 5}{1 + 5x + 3x^2}$$

on the interval [1, 4]. We use the four equispaced points $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$.
(a)

(i)

$$f(x_0) = f(1) = \frac{1^4 + 2 \cdot 1^2 + 5}{1 + 5 \cdot 1 + 3 \cdot 1^2} = \frac{8}{9} \approx 0.8889$$
$$f(x_1) = f(2) = \frac{2^4 + 2 \cdot 2^2 + 5}{1 + 5 \cdot 2 + 3 \cdot 2^2} = \frac{29}{23} \approx 1.2609$$
$$f(x_2) = f(3) = \frac{3^4 + 2 \cdot 3^2 + 5}{1 + 5 \cdot 3 + 3 \cdot 3^2} = \frac{104}{43} \approx 2.4186$$
$$f(x_3) = f(4) = \frac{4^4 + 2 \cdot 4^2 + 5}{1 + 5 \cdot 4 + 3 \cdot 4^2} = \frac{293}{69} \approx 4.2464$$

(ii)
The appropriate polynomial expansion, *g(x)*, is as follows:

$$g(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3$$

(iii)
The constraints to solve this equation are that $g(x_0) = f_0$, $g(x_1) = f_1$, $g(x_2) = f_2$, and $g(x_3) = f_3$. This gives us the following set of simultaneous equations:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

(iv) Plugging the values into the above equation yields the following:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{8}{9} \\ \frac{29}{23} \\ \frac{104}{43} \\ \frac{293}{69} \end{bmatrix}$$

Subtracting the first equation from the other equations gives us:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{8}{9} \\ \frac{77}{207} \\ \frac{592}{387} \\ \frac{695}{207} \end{bmatrix}$$

We then subtract 2 times the second equation from the third equation and divide the third equation by 2. We also subtract 3 times the second equation from the four equation gives us and divide the fourth equation by 6:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 1 & 7 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{8}{9} \\ \frac{77}{207} \\ \frac{3497}{8901} \\ \frac{232}{621} \end{bmatrix}$$

Subtracting the third equation from the fourth equation gives us this result:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{8}{9} \\ \frac{77}{207} \\ \frac{3497}{8901} \\ -\frac{515}{26703} \end{bmatrix}$$

We then use simple back-substitution to solve the system of equations as follows:

$$a_3 = -\frac{515}{26703}$$

$$a_2 = \frac{3497}{8901} - 6 \cdot (-\frac{515}{26703}) = \frac{503}{989}$$

$$a_1 = \frac{77}{207} + 7 \cdot \frac{515}{26703} - 3 \cdot \frac{503}{989} = -\frac{27205}{26703}$$

$$a_0 = \frac{8}{9} + \frac{515}{26703} - \frac{503}{989} + \frac{27205}{26703} = \frac{12625}{8901}$$

(v) Plots for *f(x)* and *g(x)* are shown in Figure 1.

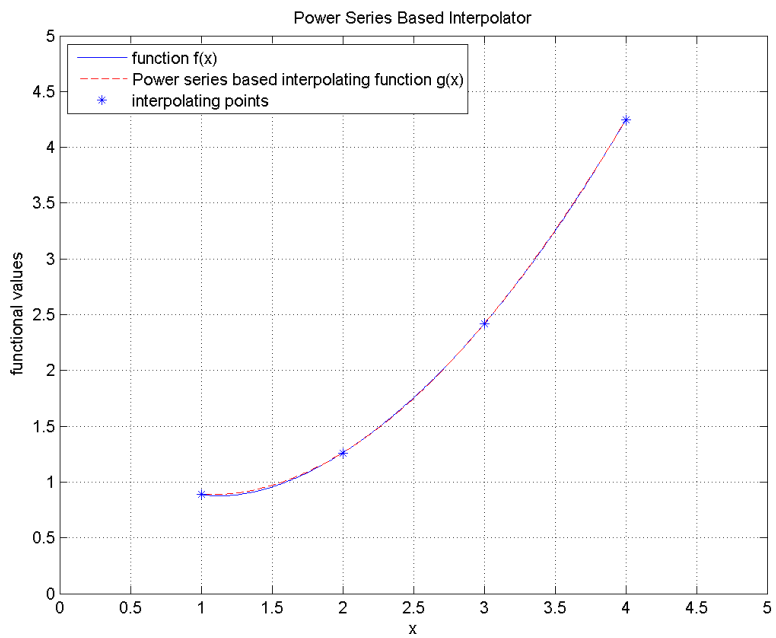


Figure 1: f(x) and power series based interpolator over the interval [1,4] using increments of 0.01.

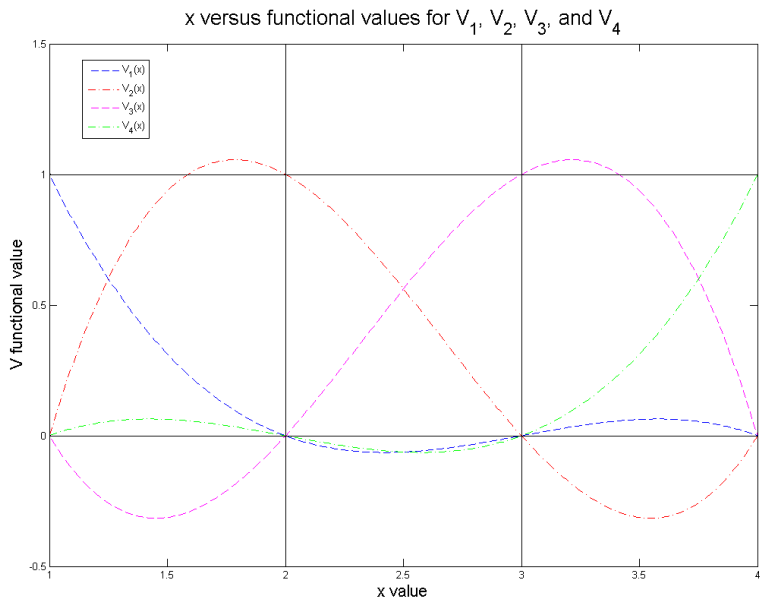


Figure 2: Lagrange Basis Functions Vi(x) over the interval [1,4] using increments of 0.01.

(b)

(i) The appropriate polynomial expansion, *g(x)*, is as follows:

$$g(x) = f_0 \cdot V_0(x) + f_1 \cdot V_1(x) + f_2 \cdot V_2(x) + f_3 \cdot V_3(x)$$

(ii)

$$V_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$V_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$V_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$V_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Plugging in values we obtain:

$$V_0(x) = -\frac{(x-2)(x-3)(x-4)}{6}$$

$$V_1(x) = \frac{(x-1)(x-3)(x-4)}{2}$$

$$V_2(x) = -\frac{(x-1)(x-2)(x-4)}{2}$$

$$V_3(x) = \frac{(x-1)(x-2)(x-3)}{6}$$

The plots for V_0, V_1, V_2, V_3 are show in Figure 2.

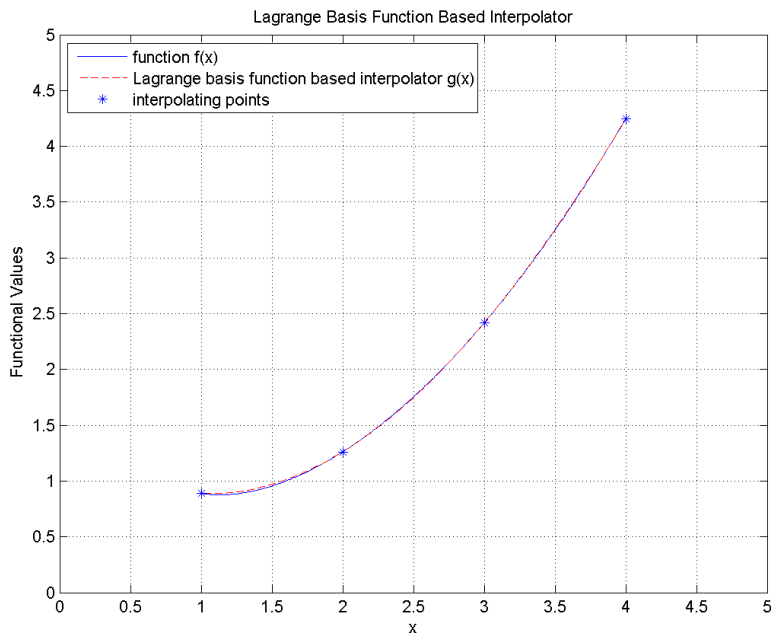


Figure 3: f(x) and Lagrange basis function based interpolator over the interval [1,4] using increments of 0.01.

(iii) The expression for g(x) is as follows:

$$g(x) = f_0 \cdot \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f_1 \cdot \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+f_2 \cdot \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f_3 \cdot \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Plugging in values for x_0, x_1, x_2, and x_3 yields the following:

$$g(x) = -\frac{8}{9} \cdot \frac{(x-2)(x-3)(x-4)}{6} + \frac{29}{23} \cdot \frac{(x-1)(x-3)(x-4)}{2} - \frac{104}{43} \cdot \frac{(x-1)(x-2)(x-4)}{2} + \frac{293}{69} \cdot \frac{(x-1)(x-2)(x-3)}{6}$$

(iv) Plots for *f(x)* and *g(x)* are shown in Figure 3.

i	f_i	Δ f_i	Δ^2 f_i	Δ^3 f_i	Δ^4 f_i
0	8/9	77/207	6994/8901	-1030/8901	93680/899001
1	29/23	1145/989	1988/2967	-50/4343	
2	104/43	5423/2967	197338/299667		
3	293/69	17327/6969			
4	680/101				

Table 1: Forward difference table.

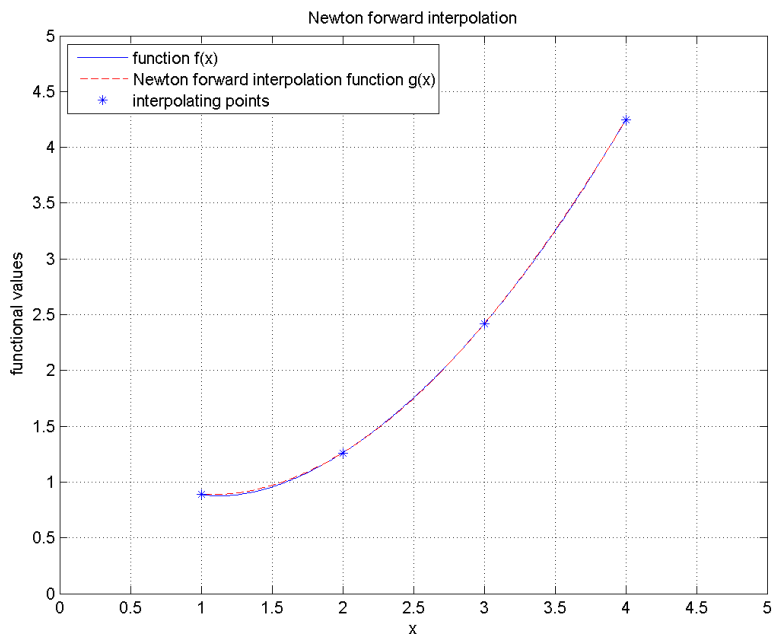


Figure 4: f(x) and Newton Forward interpolator over the interval [1,4] using increments of 0.01.

(c)

(i) Since we have four data points, the Newton Forward interpolator will be a 3rd degree polynomial. More specifically, *g(x)* is as follows:

$$g(x) = f_0 + (x - x_0)\frac{\Delta f_0}{h} + \frac{1}{2}(x - x_0)(x - x_1)\frac{\Delta^2 f_0}{h^2} + \frac{1}{6}(x - x_0)(x - x_1)(x - x_2)\frac{\Delta^3 f_0}{h^3}$$

Plugging in $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $h = 1$, and the values for the forward differences obtained in part (ii) below, we obtain the following:

$$g(x) = \frac{8}{9} + (x - 1)\frac{77/207}{1} + \frac{1}{2}(x - 1)(x - 2)\frac{6994/8901}{1^2} + \frac{1}{6}(x - 1)(x - 2)(x - 3)\frac{-1030/8901}{1^3}$$

(ii) The forward difference table can be found in Table 1. Note that we now include a fifth point since it is needed in part (e) below.

(iii) Plots for *f(x)* and *g(x)* are shown in Figure 4.

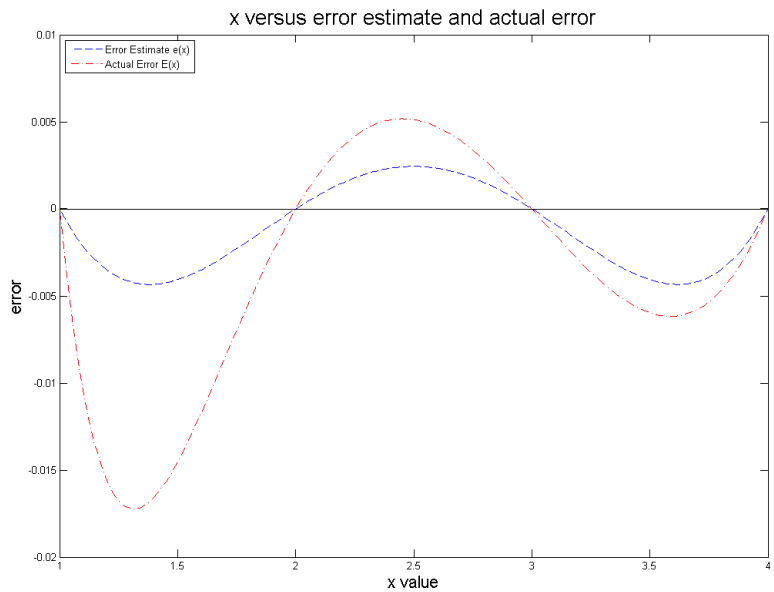


Figure 5: Error estimate e(x) and the actual error E(x) over the interval [1,4] using increments of 0.01.

(iv) The g(x) derived in steps a, b, and c appear to be the same, and we know theoretically that this is true. Moreover, the code for this problem (asgn3_prob1_c.m) sums up the errors for each of the three g's and prints them out. The program output is as follows:
Sum of the absolute values of differences
between f and g for newton = 1.773408
between f and g for lagrange = 1.773408
between f and g for power series = 1.773408
As can be seen, all three g's result in the same errors, serving as further proof that they are equivalent.

(d) e(x) can be approximated with the next term in the Newton Forward Interpolation formula:

$$e(x) = \frac{1}{4!}(x - x_0)(x - x_1)(x - x_2)(x - x_3)\frac{\Delta^4 f_0}{h^4}$$

Plugging in values we obtain:

$$e(x) = \frac{1}{24}(x - 1)(x - 2)(x - 3)(x - 4)\frac{93680/899001}{1^4}$$

We then implement this equation in a matlab function (prob_1_error.m) and plot values for this function and the actual error over the interval [1,4]. The plots are shown in Figure 5.

(e) We can add on the error term in step d to the Newton Forward interpolater in order to make it a more accurate fourth degree interpolator.

$$g(x) = \frac{8}{9} + (x - 1)\frac{77/207}{1} + \frac{1}{2}(x - 1)(x - 2)\frac{6994/8901}{1^2}$$

$$+\frac{1}{6}(x - 1)(x - 2)(x - 3)\frac{-1030/8901}{1^3} + \frac{1}{24}(x - 1)(x - 2)(x - 3)(x - 4)\frac{93680/899001}{1^4}$$

The plot for this interpolator, along with the third degree interpolator and the actual function, is shown in Figure 6. Both functions appear to be very similar. However, a summation of the absolute values of the actual error give the following result:

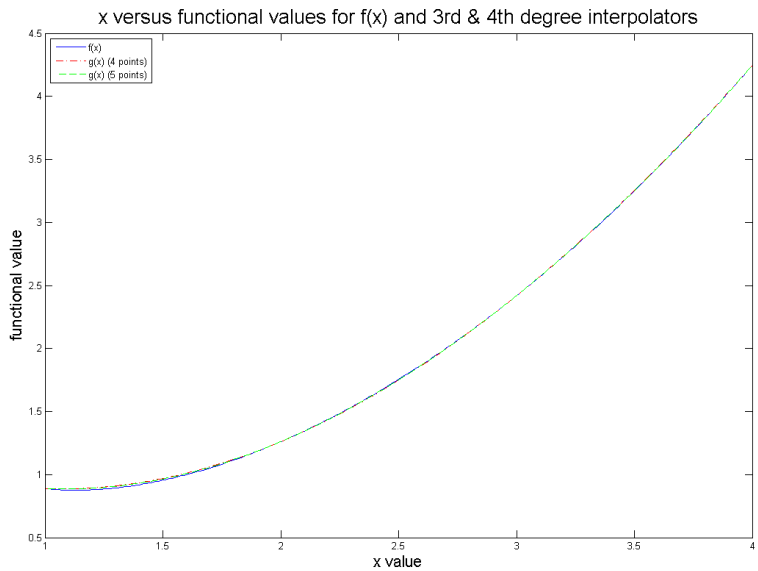


Figure 6: f(x) and the third and fourth degree Newton interpolators over the interval [1,4] using increments of 0.01.

Sum of the absolute values of differences
between f and g for newton with 4 points = 1.773408
between f and g for newton with 5 points = 1.064310
As can be seen, the fourth order interpolator does result in some increased accuracy.

Problem 2

(a) Four term Taylor series expanded about x_2:

$$f(x) = f_2 + (x - x_2) \cdot f_2^{(1)} + \frac{1}{2}(x - x_2)^2 \cdot f_2^{(2)} + \frac{1}{6}(x - x_2)^3 \cdot f_2^{(3)} + O(x - x_2)^4$$

(b) The definition of ∇ f_2 is as follows:

$$\nabla f_2 = f_2 - f_1$$

Now since $f_1 = f(x_1)$, we use the Taylor series given in (a) to express f_1 in terms of f_2 and its derivatives.

$$f_1 = f_2 + (x_1 - x_2) \cdot f_2^{(1)} + \frac{1}{2}(x_1 - x_2)^2 \cdot f_2^{(2)} + \frac{1}{6}(x_1 - x_2)^3 \cdot f_2^{(3)} + O(x_1 - x_2)^4$$

$$f_1 = f_2 - h \cdot f_2^{(1)} + \frac{1}{2} \cdot h^2 \cdot f_2^{(2)} - \frac{1}{6} \cdot h^3 \cdot f_2^{(3)} + O(h^4)$$

where $h = x_2 - x_1$. We can now substitute this expression in for f_1 in the definition of ∇ f_2:

$$\nabla f_2 = f_2 - f_2 + h \cdot f_2^{(1)} - \frac{1}{2} \cdot h^2 \cdot f_2^{(2)} + \frac{1}{6} \cdot h^3 \cdot f_2^{(3)} + O(h^4)$$

Now rearranging to solve for ${f_2}^{(1)}$, we obtain:

$$h \cdot f_2^{(1)} = \nabla f_2 + \frac{1}{2} \cdot h^2 \cdot f_2^{(2)} - \frac{1}{6} \cdot h^3 \cdot f_2^{(3)} + O(h^4)$$

$$f_2^{(1)} = \frac{\nabla f_2}{h} + \frac{1}{2} \cdot h \cdot f_2^{(2)} - \frac{1}{6} \cdot h^2 \cdot f_2^{(3)} + O(h^3)$$

(c) The definition of ∇^2 f_2 is as follows:

$$\nabla^2 f_2 = \nabla f_2 - \nabla f_1 = f_2 - f_1 - f_1 + f_0 = f_2 - 2f_1 + f_0$$

Now since $f_0 = f(x_0)$, we use the Taylor series given in (a) to express f_0 in terms of f_2 and its derivatives.

$$f_0 = f_2 + (x_0 - x_2) \cdot f_2^{(1)} + \frac{1}{2}(x_0 - x_2)^2 \cdot f_2^{(2)} + \frac{1}{6}(x_0 - x_2)^3 \cdot f_2^{(3)} + O(x_0 - x_2)^4$$

$$f_0 = f_2 - 2h \cdot f_2^{(1)} + \frac{1}{2} \cdot 4h^2 \cdot f_2^{(2)} - \frac{1}{6} \cdot 8h^3 \cdot f_2^{(3)} + O(h^4)$$

where $2h = x_2 - x_0$. We can now substitute this in for f_0 and the expression from part (b) in for f_1 in the definition of ∇^2 f_2:

$$\nabla^2 f_2 = f_2 - 2 \cdot f_2 + 2h \cdot f_2^{(1)} - 2 \cdot \frac{1}{2} \cdot h^2 \cdot f_2^{(2)} + 2 \cdot \frac{1}{6} \cdot h^3 \cdot f_2^{(3)} + O(h^4) + f_2 - 2h \cdot f_2^{(1)} + \frac{1}{2} \cdot 4h^2 \cdot f_2^{(2)} - \frac{1}{6} \cdot 8h^3 \cdot f_2^{(3)} + O(h^4)$$

Canceling out terms, we obtain:

$$\nabla^2 f_2 = h^2 \cdot f_2^{(2)} - h^3 \cdot f_2^{(3)} + O(h^4)$$

Now rearranging to solve for ${f_2}^{(2)}$, we obtain:

$$f_2^{(2)} = \frac{\nabla^2 f_2}{h^2} + h \cdot f_2^{(3)} + O(h^2)$$

(d)
The original Taylor series expansion is as follows:

$$f(x) = f_2 + (x - x_2) \cdot f_2^{(1)} + \frac{1}{2}(x - x_2)^2 \cdot f_2^{(2)} + \frac{1}{6}(x - x_2)^3 \cdot f_2^{(3)} + O(x - x_2)^4$$

We first substitute for $f_2{}^{(1)}$:

$$f(x) = f_2 + (x - x_2) \cdot \left[\frac{\nabla f_2}{h} + \frac{1}{2} \cdot h \cdot f_2^{(2)} - \frac{1}{6} \cdot h^2 \cdot f_2^{(3)} + O(h^3)\right] + \frac{1}{2}(x - x_2)^2 \cdot f_2^{(2)} + \frac{1}{6}(x - x_2)^3 \cdot f_2^{(3)} + O(x - x_2)^4$$

We then expand the equation and obtain:

$$f(x) = f_2 + \frac{(x - x_2)}{h} \cdot \nabla f_2 + \frac{x - x_2}{2} \cdot h \cdot f_2^{(2)} - \frac{x - x_2}{6} \cdot h^2 \cdot f_2^{(3)} + O(h^4) + \frac{1}{2}(x - x_2)^2 \cdot f_2^{(2)} + \frac{1}{6}(x - x_2)^3 \cdot f_2^{(3)} + O(x - x_2)^4$$

Grouping terms yields:

$$f(x) = f_2 + \frac{(x - x_2)}{h} \cdot \nabla f_2 + \frac{1}{2} \cdot f_2^{(2)} \left[(x - x_2) \cdot h + (x - x_2)^2\right]$$

$$+ \frac{1}{6} \cdot f_2^{(3)} \left[(x - x_2)^3 - (x - x_2) \cdot h^2\right] + O(x - x_2)^4$$

We can then substitue for $f_2{}^{(2)}$.

$$f(x) = f_2 + \frac{(x - x_2)}{h} \cdot \nabla f_2 + \frac{1}{2} \cdot \left[\frac{\nabla^2 f_2}{h^2} + h \cdot f_2^{(3)} + O(h^2)\right] \cdot \left[(x - x_2) \cdot h + (x - x_2)^2\right]$$

$$+ \frac{1}{6} \cdot f_2^{(3)} \left[(x - x_2)^3 - (x - x_2) \cdot h^2\right] + O(x - x_2)^4$$

We can then multiply the bracketed expressions to obtain the following:

$$f(x) = f_2 + \frac{(x - x_2)}{h} \cdot \nabla f_2 + \frac{1}{2} \cdot \nabla^2 f_2 \left[\frac{(x - x_2)}{h} + \frac{(x - x_2)^2}{h^2}\right]$$

$$+ \frac{3}{6} \cdot h^2 \cdot f_2^{(3)} \cdot (x - x_2) + \frac{3}{6} \cdot h \cdot f_2^{(3)} \cdot (x - x_2)^2 + \frac{1}{6} \cdot f_2^{(3)} \left[(x - x_2)^3 - (x - x_2) \cdot h^2\right] + O(x - x_2)^4$$

Combining the $f_2{}^{(3)}$ terms yields:

$$f(x) = f_2 + \frac{(x - x_2)}{h} \cdot \nabla f_2 + \frac{1}{2} \cdot \nabla^2 f_2 \left[\frac{(x - x_2)}{h} + \frac{(x - x_2)^2}{h^2}\right]$$

$$+ \frac{1}{6} \cdot f_2^{(3)} \left[(x - x_2)^3 + 2h^2(x - x_2) + 3h(x - x_2)^2\right] + O(x - x_2)^4$$

Now rearranging the terms in the brackets:

$$f(x) = f_2 + (x - x_2)\frac{\nabla f_2}{h} + \frac{1}{2!}(x - x_2)\left[x - (x_2 - h)\right]\frac{\nabla^2 f_2}{h^2} + \frac{1}{3!}(x - x_2)\left[(x - x_2)^2 + 3h(x - x_2) + 2h^2\right] f_2^{(3)} + O(x - x_2)^4$$

Now noting that x_2 - h = x_1, we have the following result:

$$f(x) = f_2 + (x - x_2)\frac{\nabla f_2}{h} + \frac{1}{2!}(x - x_2)(x - x_1)\frac{\nabla^2 f_2}{h^2} + \frac{1}{3!}(x - x_2)\left[(x - x_2)^2 + 3h(x - x_2) + 2h^2\right] f_2^{(3)} + O(x - x_2)^4$$

Note that the fourth term is our error term. If we had calculated $f_2{}^{(3)}$ in terms of backward difference operators, the fourth term would likely simplify down to something similar to what is seen in the class notes.

Problem 1

Consider the function:

$$f(x) = \frac{x^4 + 2x^2 + 5}{1 + 5x + 3x^2}$$

on the interval [1, 4].

a) Develop a power series based interpolator for 4 equispaced points for the given function.

   i) Evaluate $f(x_0)$, $f(x_1)$, $f(x_2)$ and $f(x_3)$ using the specified function $f(x)$.

   ii) Define the appropriate polynomial expansion, $g(x)$.

   iii) Define the constraints needed to solve for the coefficients in the generic expansion.

   iv) Solve for the coefficients and plot the interpolation formula.

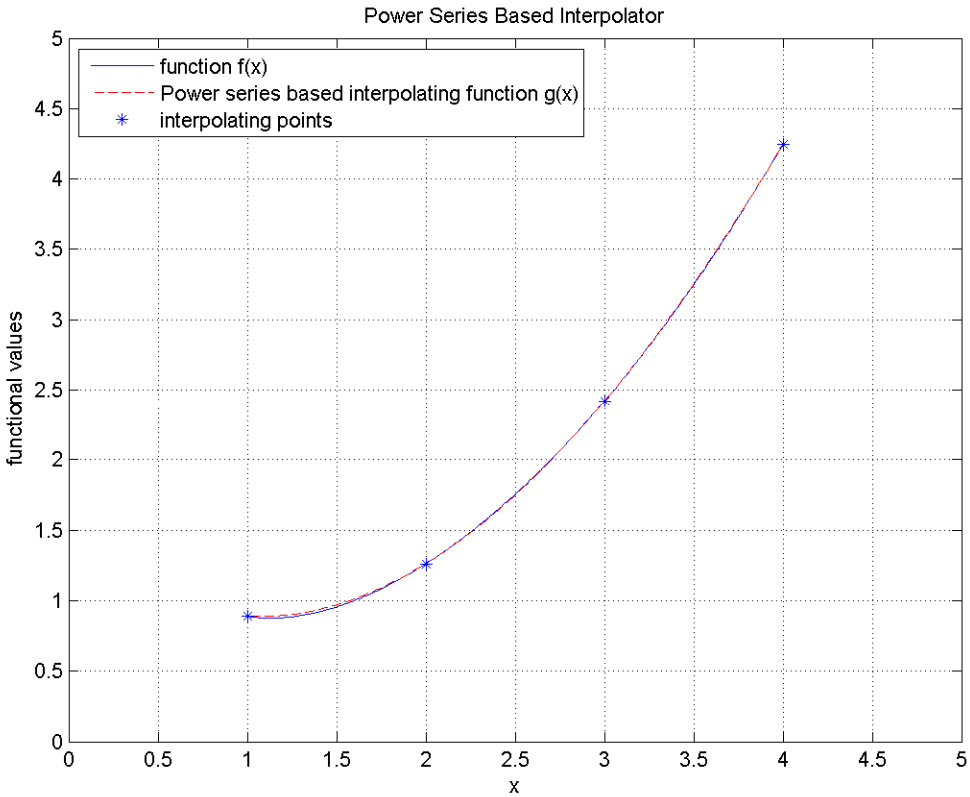
   v) Plot $f(x)$ and $g(x)$ over the interval [1, 4] using increments of 0.01. Provide labels for the axes and the curves.

```
clear all; close all;
cd ('C:\Users\chl\Documents\Class Work\Comp Methods\HW3');
%Problem 1a
% i) evaluate f(x0) f(x1)....
for x = 1:4
f(x,1)=(x^4+2*x^2+5)/(1+5*x+3*x^2);
end;clear x;

% set up matrix Xa=f
for x = 1:4
    X(x,1)=1;
    X(x,2)=x;
    X(x,3)=x^2;
    X(x,4)=x^3;
end;clear x;

%solve for A (must do by hand in HW)
A= X \ f;

%determine f(x) and g(x) on interval [1,4] using increments of 0.01.
x = 1:.01:4;S=size(x,2)
for t=1:S
fx(t,1)=(x(t)^4+2*x(t)^2+5)/(1+5*x(t)+3*x(t)^2);
gx(t,1)=A(1)+ (A(2)*x(t)) + (A(3)*x(t)^2) + (A(4)*x(t)^3);
end;clear t;
%make plot of f(x) and g(x)
plot(x,fx);hold on;plot(x,gx,'r--');plot(1:4,f,'b*')
xlabel('x');ylabel('functional values');axis([0 5 0 5]);grid on;
legend('function f(x)','Power series based interpolating function
g(x)','interpolating points','location','NW');
title('Power Series Based Interpolator')
saveas(gcf,'HW3 PR1a Power Series Based Interpolator g(x) vs
f(x).png');
```


Power Series Based Interpolator
function f(x)
Power series based interpolating function g(x)
interpolating points
functional values
x
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

b) Develop a Lagrange basis function based interpolator for 4 equispaced points for the given function.

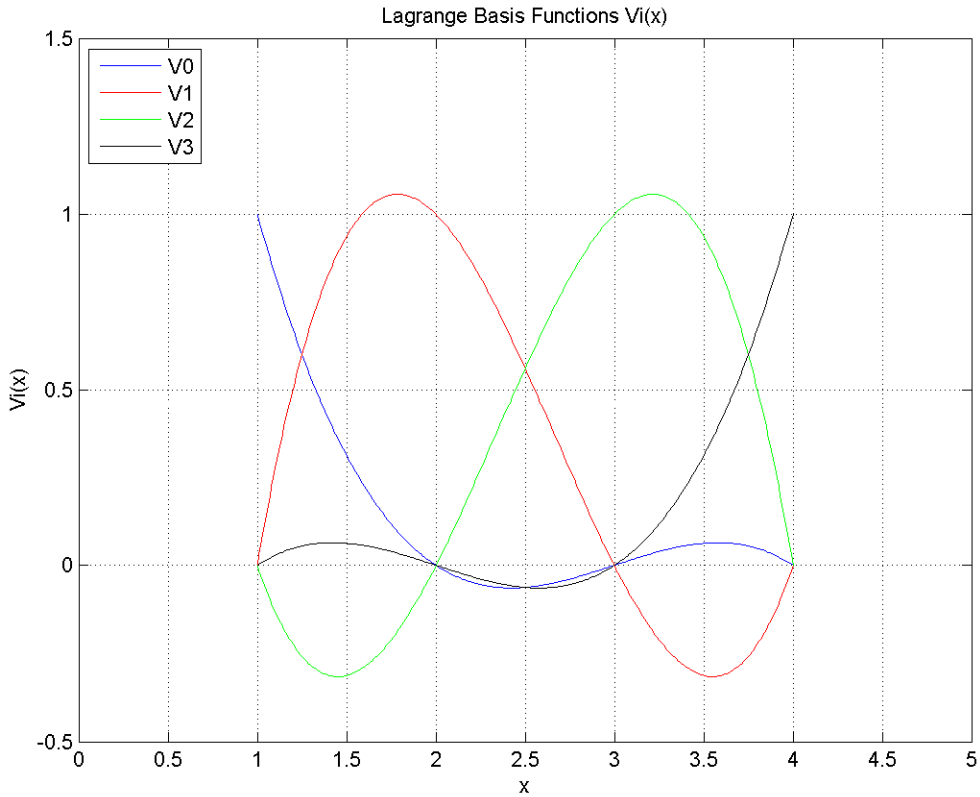
i) Define the appropriate polynomial expansion, $g(x)$.

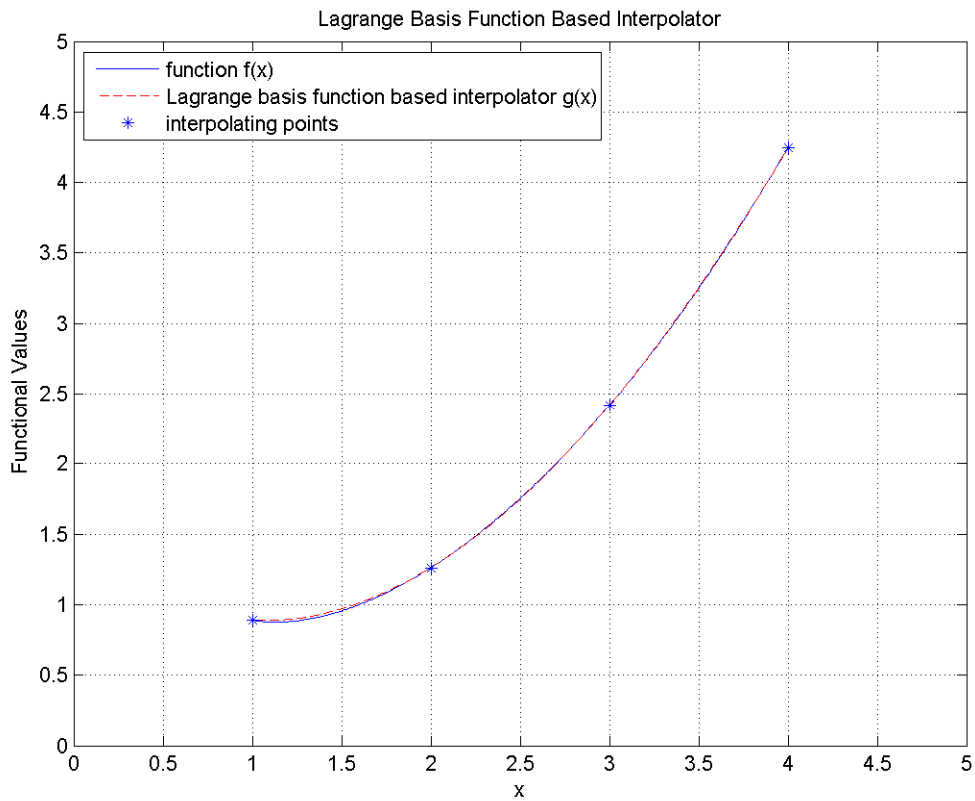
ii) Express and plot $V_0(x)$, $V_1(x)$, $V_2(x)$ and $V_3(x)$ for this problem.

iii) Write out the expression for $g(x)$ for this problem.

iv) Plot $f(x)$ and $g(x)$ over the interval [1, 4] using increments of 0.01. Provide labels for the axes and the curves.

```
%Problem 1b
clear all; close all;
%evaluate f(x0) f(x1)....y
for x = 1:4
f(x,1)=(x^4+2*x^2+5)/(1+5*x+3*x^2);
end;clear x;
% i) g(x) = f0*V0(x) + f1*V1(x) + f2*V2(x) + f3*V3(x)
% ii)
% V0(x) = (x-x1)(x-x2)(x-x3)/(x0-x1)(x0-x2)(x0-x3)
% V1(x) = (x-x0)(x-x2)(x-x3)/(x1-x0)(x1-x2)(x1-x3)
% V2(x) = (x-x0)(x-x1)(x-x3)/(x2-x0)(x2-x1)(x2-x3)
% V3(x) = (x-x0)(x-x1)(x-x2)/(x3-x0)(x3-x1)(x3-x2)
% where:
x0=1;x1=2;x2=3;x3=4;
x = 1:.01:4; %interval that Vi, f(x) and g(x) will be plotted on
%calculate and plot the Vi for i 0 to 3
for t=1:301
V0(t) = ((x(t)-x1)*(x(t)-x2)*(x(t)-x3))/((x0-x1)*(x0-x2)*(x0-x3));
V1(t) = ((x(t)-x0)*(x(t)-x2)*(x(t)-x3))/((x1-x0)*(x1-x2)*(x1-x3));
V2(t) = ((x(t)-x0)*(x(t)-x1)*(x(t)-x3))/((x2-x0)*(x2-x1)*(x2-x3));
V3(t) = ((x(t)-x0)*(x(t)-x1)*(x(t)-x2))/((x3-x0)*(x3-x1)*(x3-x2));
end
plot(x,V0); hold on; plot(x,V1,'r-');plot(x,V2,'g-');plot(x,V3,'k-')
xlabel('x');ylabel('V Funcitonal Values');axis([0 5 -.5 1.5]);
grid on;legend('V0(x)','V1(x)','V2(x)','V3(x)','location','NW');
title('Lagrange Basis Functions Vi(x)');
saveas(gcf,'HW3 PR1b ii - Lagrange Basis Function Based
Interpolator.png');
close all;
%
% iii)
% g(x) = f0*V0(x) + f1*V1(x) + f2*V2(x) + f3*V3(x)
% Calculate g(x) by adding up the the V functions and multiplying each
by f
gx=f(1)*V0+f(2)*V1+f(3)*V2+f(4)*V3;
% Calculate f(x) for interval 1:01:4 (x)
for t=1:301
fx(t)=(x(t)^4+2*x(t)^2+5)/(1+5*x(t)+3*x(t)^2);
end
%make plot of f(x) and g(x)
plot(x,fx);hold on;plot(x,gx,'r--');plot(1:4,f,'b*');
xlabel('x');ylabel('Functional Values');axis([0 5 0 5]);grid on;
legend('function f(x)','Lagrange basis function based interpolator
g(x)','interpolating points','location','NW');
title('Lagrange Basis Function Based Interpolator')
saveas(gcf,'HW3PR1biv.png');
```


Lagrange Basis Functions Vi(x)
V0
V1
V2
V3
Vi(x)
x
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


Lagrange Basis Function Based Interpolator
function f(x)
Lagrange basis function based interpolator g(x)
interpolating points
Functional Values
x
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

c) Develop a Newton Forward Interpolation based interpolator for 4 equispaced points for the given function.

i) Write down the appropriate Newton Forward interpolator, $g(x)$.

ii) Develop a forward difference table.

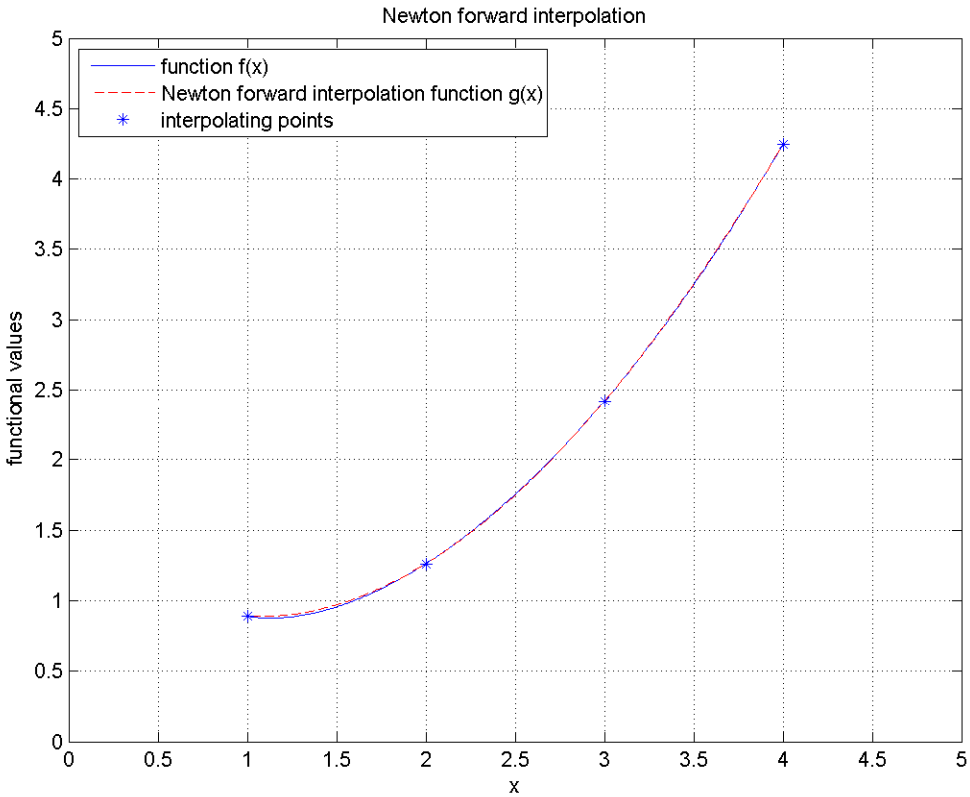
iii) Plot $f(x)$ and $g(x)$ over the interval [1, 4] using increments of 0.01. Provide labels for the axes and the curves.

iv) How do the $g(x)$ derived in steps a, b and c compare?

```
%Problem 1c
%Newton Forward Interpolation
clear all; close all;
%evaluate f(x0) f(x1)....
for x = 1:5
f(x,1)=(x^4+2*x^2+5)/(1+5*x+3*x^2);
end;clear x;
%i)
%ii) create forward difference table as on page 7.4 of the notes
%second column del^1 f
f(1,2)=f(2,1)-f(1,1);
f(2,2)=f(3,1)-f(2,1);
f(3,2)=f(4,1)-f(3,1);
f(4,2)=f(5,1)-f(4,1);%add to table for 1d, and 1e
%third column  del^2 f
f(1,3)=f(2,2)-f(1,2);
f(2,3)=f(3,2)-f(2,2);
f(3,3)=f(4,2)-f(3,2);%add to table for 1d, and 1e
%fourth column del^3 f
f(1,4)=f(2,3)-f(1,3);
f(2,4)=f(3,3)-f(2,3);%add to table for 1d, and 1e
%fifth colume del^4 f  (for problem 1d where you need the 5th point)
f(1,5)=f(2,4)-f(1,4);
% iii) plot f(x) and g(x) over the interval
x0=1;x1=2;x2=3;x3=4;x4=5;
x = 1:.01:4; %interval that Vi, f(x) and g(x) will be plotted on
h=1; %incrememnt between points
% compute g(x) as on p 7.16 of the notes, for interval [1:.01:4]
% again...compute f(x) for interval [1:.01:4]
for t=1:301
gx(t)=f(1,1)+(x(t)-x0)*(f(1,2)/h) + (1/factorial(2))*(x(t)-x0)*(x(t)-
x1)*(f(1,3)/h*h) + (1/factorial(3))*(x(t)-x0)*(x(t)-x1)*(x(t)-
x2)*(f(1,4)/h*h*h);
fx(t)=(x(t)^4+2*x(t)^2+5)/(1+5*x(t)+3*x(t)^2);
end

plot(x,fx);hold on;plot(x,gx,'r--');plot(1:4,f(1:4,1),'b*');
xlabel('x');ylabel('functional values');axis([0 5 0 5]);grid on;
legend('function f(x)','Newton forward interpolation function
g(x)','interpolating points','location','NW');
title('Newton forward interpolation')
saveas(gcf,'HW3 PR1c iii Newton Forward Interpolation Based
Interpolator.png');

% iv) g(x) looks the same for all three methods...
```

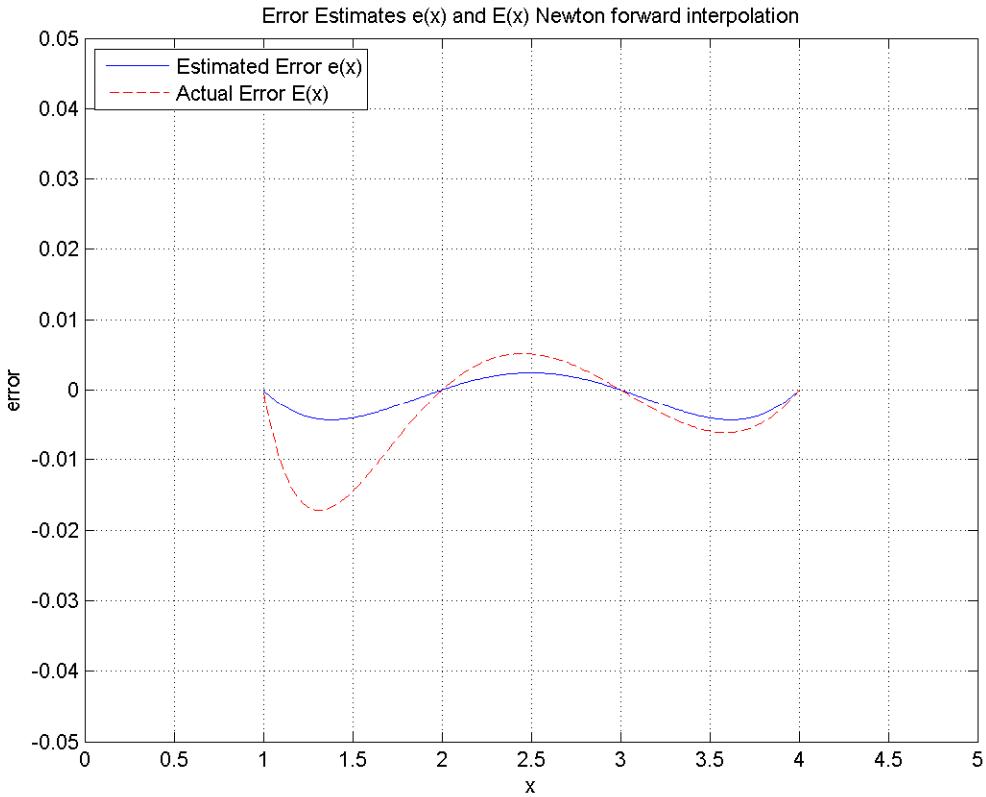

Newton forward interpolation
function f(x)
Newton forward interpolation function g(x)
interpolating points
functional values
x
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

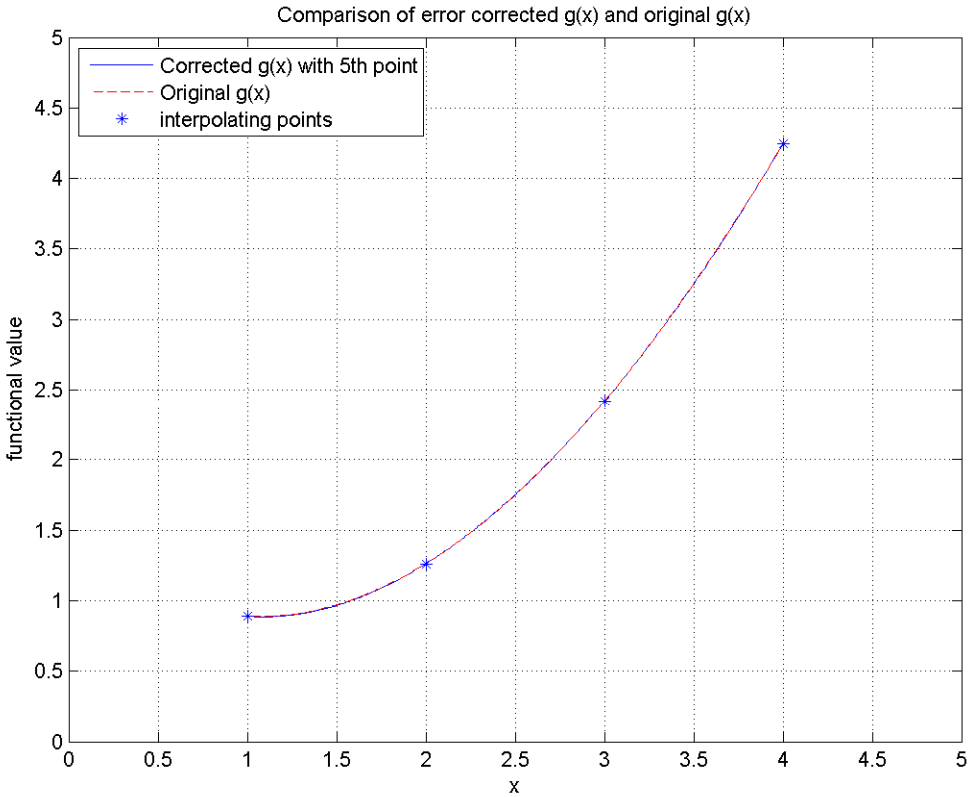
d) Estimate the error associated with $g(x)$, $e(x)$ using the next term in the Newton Forward interpolation formula. Then evaluate the actual error between $f(x)$ and $g(x)$, $E(x)$ over the interval [1, 4] using increments of 0.01. Plot both $e(x)$ and $E(x)$ over the interval. Provide labels for the axes and the curves.

e) How can you use the error term in step d, to improve the Newton Forward interpolater established in step c? Plot the now 5 point fourth degree interpolator, comparing it to the 4 point cubic interpolator established in Step c.

```
%Problem 1d
for t=1:301
ex(t)=(((x(t)-x0)*(x(t)-x1)*(x(t)-x2)*(x(t)-
x3))/(factorial(4)))*(f(1,5)/(h*h*h*h*h));
end
Ex=fx-gx;
plot(x,ex);hold on;plot(x,Ex,'r--');
xlabel('x');ylabel('error');axis([0 5 -0.05 .05]);grid on;
legend('Estimated Error e(x)','Actual Error E(x)','location','NW');
title('Error Estimates e(x) and E(x) Newton forward interpolation')
saveas(gcf,'HW3 PR1d iii Newton Forward Interpolation Based
Interpolator Error Estimates.png');


%Problem 1e
% to improve the Newton Forward interpolater established in step c, you
can add the error term to g(x)
gx_corrected=gx+ex;
plot(x,gx_corrected);hold on;plot(x,gx,'r--');plot(1:4,f(1:4,1),'b*');
xlabel('x');ylabel('functional value');axis([0 5 0 5]);grid on;
legend('Corrected g(x) with 5th point','Original g(x)','interpolating
points','location','NW');
title('Comparison of error corrected g(x) and original g(x)')
saveas(gcf,'HW3 PR1e Newton Forward Interpolation Error Correction with
5th point.png');
```


Error Estimates e(x) and E(x) Newton forward interpolation
Estimated Error e(x)
Actual Error E(x)
error
x
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


Comparison of error corrected g(x) and original g(x)
Corrected g(x) with 5th point
Original g(x)
interpolating points
functional value
x
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5